

## 1 Introduction

In circuit analysis one often needs to determine the sensitivity of a circuit voltage or current to changes in a circuit element. While this can be done using numerical techniques, such as the sensitivity analysis in SPICE[1], a better insight can sometimes be obtained by analytical methods. Sensitivity analysis is essentially taking a partial derivative of the voltage or current with respect to the circuit element parameter. Unless the derivative is to be evaluated by a numerical method, the circuit must be solved with the parameter whose sensitivity is required left as an algebraic variable, not as a number. For complicated circuits, this may require a difficult solution of simultaneous equations in algebraic form followed by a tedious evaluation of the partial derivative. An alternative is to do implicit differentiation, substitute the numerical values into this result, and then do the solution for the partial derivative by numerical solution of the simultaneous equations.

The widely used circuit analysis text by Nilsson and Riedel[2] discusses the sensitivity analysis problem, but only demonstrates its solution by solving for the circuit variables first. The following derivation and example provides a useful supplement to this text when additional emphasis is placed on the sensitivity topic.

## 2 Implicit differentiation

In calculus, implicit differentiation is performed in the following manner. Suppose there are  $N$  dependent variables,  $\{y_j, j = 1 \dots N\}$ , and  $M$  independent variables,  $\{x_i, i = 1 \dots M\}$ . Then if one has  $N$  implicit equations,  $\{f_k(x_i, y_j) = 0, k = 1 \dots N\}$ , these  $N$  equations may have total differentials taken to obtain  $N$  equations which are linear in the differentials of the dependent and independent variables and also depend on the dependent and independent variables:

$$df_k = \sum_{i=1}^M \frac{\partial f_k}{\partial x_i} dx_i + \sum_{j=1}^N \frac{\partial f_k}{\partial y_j} dy_j = 0, \quad k = 1 \dots N. \quad (1)$$

This set of equations may be solved for the  $N$  differentials  $dy_j$  to obtain

$$dy_j = \sum_{i=1}^M \frac{\partial y_j}{\partial x_i} dx_i, \quad j = 1 \dots N, \quad (2)$$

from which the desired partial derivatives may be extracted as the coefficients of the  $dx_i$ .

## 3 Application to electric circuits

In a circuit, the dependent variables are the branch currents and branch potential differences. These may be obtained as combinations of the variables used in loop current or node voltage methods of analysis. The independent variables are the circuit element parameters, such as resistance, inductance, value of a current source, *etc.*

When time dependent circuit analysis is done, the time,  $t$ , is also a variable and the circuit equations are differential equations. The technique describe here may still be applied if a state-variable approach is used to obtain the circuit equations. The first order time derivatives of the state variables are treated as separate dependent variables.

When phasor analysis is done, the complex variable partial derivatives may be obtain. However, to determine sensitivity to frequency variation as well as individual reactive component variation, the impedances and admittances should be replaced by their real and imaginary parts, given in terms of resistance, inductance, and capacitance.

When dependent sources are present, they need to have their values expressed in terms of the other circuit parameters before the analysis is begun.

With such of these special situations taken into account, one selects an appropriate set of independent and dependent variables and expresses the equations in matrix form:

$$[A][Y] = [B][X]. \quad (3)$$

In eq.(3), the matrix  $[A]$  must have an inverse if the system is to be solved.

Next, take the total differential of this matrix equation, eq.(3):

$$(d[A])[Y] + [A](d[Y]) = (d[B])[X] + [B](d[X]). \quad (4)$$

In eq.(4) the differentials of the matrices are taken element by element.

With the system of equations of eq.(4) set up, the next step is to replace the circuit element parameters by their numerical values at the nominal operating point. Following this replacement, eq.(4) is multiplied by the inverse of matrix  $[A]$  to obtain the explicit solution for the total differentials of the independent variables:

$$d[Y] = \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_N \end{bmatrix} = [A]^{-1}[B](d[X]) + [A]^{-1}(d[B])[X] - [A]^{-1}(d[A])[Y]. \quad (5)$$

If desired, one may substitute  $[Y] = [A]^{-1}[B][X]$  in the last term of eq.(5). Then inspection of eq.(5) yields the partial derivatives of the  $y_j$  with respect to all the circuit element values and the independent source values (the  $x_i$ ) as the multipliers of the corresponding differentials.

When only a few of the partial derivatives are wanted, the solutions may be simplified by setting the differentials of parameters not to be used to zero at the stage of eq.(4).

## 4 Example

The use of implicit differentiation to find a circuit sensitivity is illustrated in the following example of a resistive circuit solved by the mesh current method.

The circuit of Fig. 1 has three mesh currents and requires three Kirchoff voltage law equations for its solution. They may be written in algebraic form as

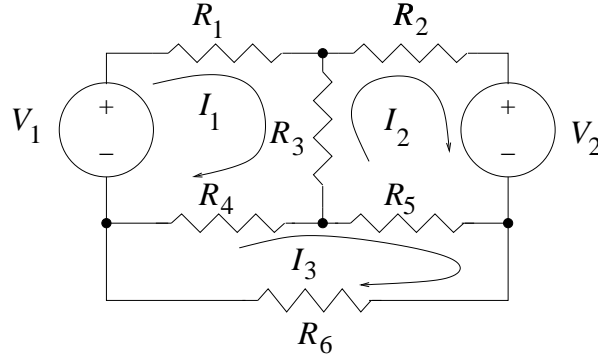


Figure 1: A simple resistive circuit with three mesh currents.

$$\begin{bmatrix} (R_1 + R_3 + R_4) & -R_3 & -R_4 \\ -R_3 & (R_3 + R_2 + R_5) & -R_5 \\ -R_4 & -R_5 & (R_4 + R_5 + R_6) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \\ 0 \end{bmatrix}. \quad (6)$$

Next, take the differential of eq.(6) to obtain

$$\begin{bmatrix} (dR_1 + dR_3 + dR_4) & -dR_3 & -dR_4 \\ -dR_3 & (dR_3 + dR_2 + dR_5) & -dR_5 \\ -dR_4 & -dR_5 & (dR_4 + dR_5 + dR_6) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} (R_1 + R_3 + R_4) & -R_3 & -R_4 \\ -R_3 & (R_3 + R_2 + R_5) & -R_5 \\ -R_4 & -R_5 & (R_4 + R_5 + R_6) \end{bmatrix} \begin{bmatrix} dI_1 \\ dI_2 \\ dI_3 \end{bmatrix} = \begin{bmatrix} dV_1 \\ -dV_2 \\ 0 \end{bmatrix}. \quad (7)$$

Eq.(7) may be solved for the differentials of the mesh currents as

$$\begin{bmatrix} dI_1 \\ dI_2 \\ dI_3 \end{bmatrix} = \begin{bmatrix} (R_1 + R_3 + R_4) & -R_3 & -R_4 \\ -R_3 & (R_3 + R_2 + R_5) & -R_5 \\ -R_4 & -R_5 & (R_4 + R_5 + R_6) \end{bmatrix}^{-1} \begin{bmatrix} dV_1 \\ -dV_2 \\ 0 \end{bmatrix} - \begin{bmatrix} (R_1 + R_3 + R_4) & -R_3 & -R_4 \\ -R_3 & (R_3 + R_2 + R_5) & -R_5 \\ -R_4 & -R_5 & (R_4 + R_5 + R_6) \end{bmatrix}^{-1} \times \begin{bmatrix} (dR_1 + dR_3 + dR_4) & -dR_3 & -dR_4 \\ -dR_3 & (dR_3 + dR_2 + dR_5) & -dR_5 \\ -dR_4 & -dR_5 & (dR_4 + dR_5 + dR_6) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad (8)$$

One now leaves the analytic solution and substitutes numerical values for the nominal values of the independent variables of the system. Suppose, for this example, that  $R_1 = 4\Omega$ ,  $R_2 = 2\Omega$ ,  $R_3 = 5\Omega$ ,  $R_4 = 1\Omega$ ,  $R_5 = 2\Omega$ ,  $R_6 = 3\Omega$ ,  $V_1 = 8\text{ V}$ , and  $V_2 = 7\text{ V}$ . The mesh equations now become

$$\begin{bmatrix} 10 & -5 & -1 \\ -5 & 9 & -2 \\ -1 & -2 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 0 \end{bmatrix} \text{ A} \quad (9)$$

with solution

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0.155763 & 0.099688 & 0.059190 \\ 0.099688 & 0.183801 & 0.077882 \\ 0.059190 & 0.077882 & 0.202492 \end{bmatrix} \begin{bmatrix} 8 \\ -7 \\ 0 \end{bmatrix} \text{ A} = \begin{bmatrix} 0.548287 \\ -0.489097 \\ -0.0716518 \end{bmatrix} \text{ A} \quad (10)$$

Suppose one wishes to find the sensitivity of the current in the  $5\Omega$  resistor with respect to changes in the  $3\Omega$  resistor. One only needs the partial derivatives of  $I_1$  and  $I_2$  with respect to  $R_6$ . Thus, in eq.(9), all the differentials on the right hand side may be set to zero except for  $dR_6$ . This simplifies the equations to

$$\begin{bmatrix} dI_1 \\ dI_2 \\ dI_3 \end{bmatrix} = - \begin{bmatrix} 0.155763 & 0.099688 & 0.059190 \\ 0.099688 & 0.183801 & 0.077882 \\ 0.059190 & 0.077882 & 0.202492 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & dR_6 \end{bmatrix} \begin{bmatrix} 0.548287 \\ -0.489097 \\ -0.0716518 \end{bmatrix} \text{ A}/\Omega \quad (11)$$

Finally, one has

$$\begin{bmatrix} dI_1 \\ dI_2 \\ dI_3 \end{bmatrix} = dR_6 \begin{bmatrix} 0.0042410 \\ 0.0055803 \\ 0.0145088 \end{bmatrix} \text{ A}/\Omega \quad (12)$$

from which we find that the partial derivative of the current in the  $5\Omega$  resistor with respect to changes in the value of the  $3\Omega$  resistor is  $-0.0013393 \text{ A}/\Omega$ .

## 5 Conclusions

The use of implicit differentiation reduces the effort at finding explicit sensitivity formulas, in the form of partial derivatives, while still retaining insights that may be obscured by purely numerical analysis. This technique deserves a place in the circuit theory curriculum.

## References

- [1] T. Quarles, A. R. Newton, D. O. Pederson, A. Sangiovanni-Vincentelli, *SPICE3 Version 3f3 User's Manual*, Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, California, May 1993.
- [2] James W. Nilsson and Susan A. Riedel, *Electric Circuits, 7th ed.*, Pearson Education, Inc., Upper Saddle River, New Jersey, 2005.